AMALGAMATED FREE PRODUCTS OF C*-BUNDLES

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ABSTRACT. Given two unital continuous C*-bundles A and B over the same compact Hausdorff base space X, we study the continuity properties of their different amalgamated free products over $C(X)$.

In memory of Gert Pedersen

1. Introduction

Tensor products of C*-bundles have been much studied over the last decades (see for example [30], [20], [5], [22], [21], [1], [23], [11]). One of the main results was obtained by Kirchberg and Wassermann who gave in [22] a characterization of the exactness (respectively the nuclearity) of the C*-algebra of sections A of a continuous bundle of C*-algebras over a compact Hausdorff space X through the equivalence between the following conditions α_e) and β_e) (respectively α_n) and β_n)):

α_e) *The C*-bundle A is an exact C*-algebra.*

β_e) *For all continuous C*-bundle B over a compact Hausdorff space Y, the minimal C*-tensor product $A \overset{m}{\otimes} B$ is a continuous C*-bundle over $X \times Y$ with fibres $A_x \overset{m}{\otimes} B_y$.*

α_n) *The C*-bundle A is a nuclear C*-algebra.*

β_n) *For all continuous C*-bundle B over a compact Hausdorff space Y, the maximal C*-tensor product $A \overset{M}{\otimes} B$ is a continuous C*-bundle over $X \times Y$ with fibres $A_x \overset{M}{\otimes} B_y$.*

Remark 1.1. In [22], the authors add to condition β_e) the assumption that all the fibres A_x ($x \in X$) are exact. But this is automatically satisfied (see [11, Prop 3.3]).

The case when we restrict our attention to fibrewise tensor products was then extensively studied in [11]. The two first assertions (respectively the two last ones) are indeed equivalent to the following assertion γ_e) (respectively γ_n)) introduced in [5], in case the compact Hausdorff space X is perfect and second countable.

γ_e) *For all continuous $C(X)$-algebra B, the smallest completion $A \underset{C(X)}{\overset{m}{\otimes}} B$ of the algebraic tensor product $A \underset{C(X)}{\odot} B$ amalgamated over $C(X)$ is a continuous C*-bundle over X with fibres $A_x \overset{m}{\otimes} B_x$.*

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But there are also other canonical amalgamated products over $C(X)$, such as the completions considered by Pedersen ([26]) and Voiculescu ([32]) of the algebraic amalgamated free product $A \underset{C(X)}{\circledast} B$ of two unital continuous C*-bundles A and B over the same compact Hausdorff space X. The point of this paper is to study whether analogous continuity properties hold (or not) for these amalgamated free products.

More precisely, we start in §2 by fixing our notations and extending a few results available for $C(X)$-algebras to the framework of the operator systems which naturally appear when dealing with free products of $C(X)$-algebras amalgamated over $C(X)$. We show in §3 that the full amalgamated free products are always continuous (Theorem 3.7) and we prove in §4 that the exactness of the C*-algebra A is sufficient to ensure the continuity of the reduced ones (Theorem 4.1). In particular, this implies that any separable continuous C*-bundle over a compact Hausdorff space X admits a $C(X)$-linear embedding into a C*-algebra with Hausdorff primitive ideal space X.

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2. Preliminaries

We recall in this section a few basic definitions and constructions related to the theory of C*-bundles.

Let us first fix a few notations for operators acting on Hilbert C*-modules ([4, §13]).

Definition 2.1. Let B be a C*-algebra and E a Hilbert B-module.
– For all $\zeta_1, \zeta_2 \in E$, we define the rank 1 operator θ_{ζ_1,ζ_2} acting on the Hilbert B-module E by the relation

$$\theta_{\zeta_1,\zeta_2}(\zeta) = \zeta_1\langle\zeta_2, \zeta\rangle \quad (\zeta \in E). \tag{2.1}$$

– The closed linear span of these operators is the C*-algebra $\mathcal{K}_B(E)$ of *compact* operators acting on the Hilbert B-module E.
– The multiplier C*-algebra of $\mathcal{K}_B(E)$ is (isomorphic to) the C*-algebra $\mathcal{L}_B(E)$ of continuous adjointable B-linear operators acting on E
– In case $B = \mathbb{C}$, then E is a Hilbert space and we simply denote by $\mathcal{L}(E)$ and $\mathcal{K}(E)$ the C*-algebras $\mathcal{L}_{\mathbb{C}}(E)$ and $\mathcal{K}_{\mathbb{C}}(E)$. A basic example is the separable Hilbert space $\ell_2(\mathbb{N})$ of complex valued sequences $(a_i)_{i\in\mathbb{N}}$ which satisfy $\|(a_i)\|^2 = \sum_i |a_i|^2 < \infty$.

Let X be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous functions on X with values in the complex field $\mathbb{C}$.

Definition 2.2. A $C(X)$-algebra is a C*-algebra A endowed with a unital $*$-homomorphism from $C(X)$ to the centre of the multiplier C*-algebra $\mathcal{M}(A)$ of A.

For all $x \in X$, we denote by $C_x(X)$ the ideal of functions $f \in C(X)$ satisfying $f(x) = 0$. We denote by A_x the quotient of A by the *closed* ideal $C_x(X)A$ and by a_x the image of an element $a \in A$ in the *fibre* A_x. Then the function

$$x \mapsto \|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\} \tag{2.2}$$

is upper semi-continuous by construction. The $C(X)$-algebra is said to be *continuous* (or to be a continuous C*-bundle over X in [15], [6], [22]) if the function $x \mapsto \|a_x\|$ is actually continuous for all element a in A.

Examples 2.3. Given a C*-algebra D, the spatial tensor product $A = C(X) \otimes D = C(X; D)$ admits a canonical structure of continuous $C(X)$-algebra with constant fibre $A_x \cong D$. Thus, if A' is a C*-subalgebra of A stable under multiplication with $C(X)$, then A' also defines a continuous $C(X)$-algebra. This is especially the case for separable exact continuous $C(X)$-algebras: they always admit a $C(X)$-embedding in the constant $C(X)$-algebra $C(X; \mathcal{O}_2)$, where $\mathcal{O}_2$ is the Cuntz C*-algebra ([7]).

Definition 2.4. ([5]) Given a continuous $C(X)$-algebra B, a *continuous field of faithful representations* of a $C(X)$-algebra A on B is a $C(X)$-linear map π from A to the multiplier C*-algebra $\mathcal{M}(B)$ of B such that, for all $x \in X$, the induced representation π_x of the fibre A_x in $\mathcal{M}(B_x)$ is faithful.

Note that the existence of such a continuous field of faithful representations π implies that the $C(X)$-algebra A is continuous since the function

$$x \mapsto \|a_x\| = \|\pi_x(a_x)\| = \|\pi(a)_x\| = \sup\{\|(\pi(a)b)_x\|, b \in B \text{ such that } \|b\| \leq 1\} \tag{2.3}$$

is lower semi-continuous for all $a \in A$.

Conversely, any *separable* continuous $C(X)$-algebra A admits a continuous field of faithful representations. More precisely, there always exists a unital positive $C(X)$-linear map $\varphi : A \to C(X)$ such that all the induced states φ_x on the fibres A_x are faithful ([6]). By the Gel'fand-Naimark-Segal (GNS) construction this gives a continuous field of faithful representations of A on the continuous $C(X)$-algebra of compact operators $\mathcal{K}_{C(X)}(E)$ on the Hilbert $C(X)$-module $E = L^2(A, \varphi)$.

These constructions admit a natural extension to the framework of operator systems. Indeed, for all Banach space V with a unital contractive homomorphism from $C(X)$ into the bounded linear operators on V, one can define the fibres $V_x = V/C_x(X)V$ and the projections $v \in V \mapsto v_x := v + C_x(X)\,V \in V_x$ ([15], [10, §2.3]). Then the following $C(X)$-linear version of Ruan's characterization of operator spaces holds.

Proposition 2.5. *Let W be a separable operator system which is a unital $C(X)$-module such that, for all positive integer n and all w in $M_n(W)$, the map $x \mapsto \|w_x\|$ is continuous.*

(i) *Every unital completely positive map ϕ from a fibre W_x to $M_n(\mathbb{C})$ admits a $C(X)$-linear unital completely positive extension $\varphi : W \to M_n(C(X))$.*
(ii) *There exist a Hilbert $C(X)$-module E and a $C(X)$-linear map $\Phi : W \to \mathcal{L}_{C(X)}(E)$ such that for all $x \in X$, the induced map from W_x to $\mathcal{L}(E_x)$ is completely isometric.*

Proof. (i) Let $\zeta_n \in \mathbb{C}^n \otimes \mathbb{C}^n$ be the unit vector $\zeta_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$. Then the state $w \mapsto \langle \zeta_n, (id_n \otimes \phi)(w)\, \zeta_n \rangle$ on $M_n(\mathbb{C}) \otimes W_x \cong M_n(W_x)$ admits a $C(X)$-linear unital positive extension to $M_n(W)$ ([6], [10]). Thus, there is a $C(X)$-linear unital completely positive (u.c.p.) map $\varphi : W \to M_n(C(X))$ with $\varphi_x = \phi$ and $\|\varphi\|_{cb} = \|\phi\|_{cb}$ ([25], [34]).

(ii) The proof is the same as the one of Theorem 2.3.5 in [19]. Indeed, given a point $x \in X$ and an element $w \in M_k(W)$, there exists, by lemma 2.3.4 and proposition 2.2.2 of [19], a u.c.p. map φ_x from W_x to $M_k(\mathbb{C})$, such that

$$\|(\imath_k \otimes \varphi_x)(w_x)\| = \|w_x\|\,,$$

and one can extend φ_x to a $C(X)$-linear u.c.p. map $\varphi : W \to M_k(C(X))$ by part (i).

For all $n \geq 1$, let $\mathfrak{s}_n$ be the set of completely contractive $C(X)$-linear maps from W to $M_n(C(X))$ and let $\mathfrak{s} = \oplus_n \mathfrak{s}_n$. Then the map $w \mapsto (\varphi(w))_{\varphi \in \mathfrak{s}}$ defines an appropriate $C(X)$-linear completely isometric representation of W. ☐

Remark 2.6. Let $\{M_n(W), \|\,.\,\|_n\}$ be a separable operator system such that W is a continuous $C(X)$-module. Then the formula

$$\|w\|_n^{\sim} = \sup\{\|\langle \xi \otimes 1, (1_n \otimes w)\, \eta \otimes 1\rangle\|; \xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^n \text{ unit vectors}\}$$

for $w \in M_n(W)$ defines an operator system structure on W satisfying the hypotheses of Proposition 2.5.

The Proposition 2.5 also induces the following $C(X)$-linear Wittstock extension:

Corollary 2.7. *Let X be a compact Hausdorff space, A a separable unital continuous $C(X)$-algebra and let V be a $C(X)$-submodule of A.*

Then any completely contractive map ϕ from a fibre V_x to $M_{k,l}(\mathbb{C})$ admits a $C(X)$-linear completely contractive extension $\varphi : V \to M_{k,l}(C(X))$.

Proof. Let W be the $C(X)$-linear operator subsystem $\begin{bmatrix} C(X) & V \\ V^* & C(X) \end{bmatrix}$ of $M_2(A)$ and let $\tilde{\phi}$ be the unital completely positive map from the fibre W_x to $M_{k+l}(\mathbb{C})$ given by

$$\tilde{\phi}(\begin{bmatrix} \alpha & v'_x \\ v^*_x & \beta \end{bmatrix}) = \begin{bmatrix} \alpha & \phi(v'_x) \\ \phi(v_x)^* & \beta \end{bmatrix}.$$

Let $\zeta = (k+l)^{-1/2} \sum_i e_i \otimes e_i \in \mathbb{C}^{k+l} \otimes \mathbb{C}^{k+l}$. Then the associated state $\psi(d) = \langle \zeta, (\tilde{\phi} \otimes \imath)(d)\zeta \rangle = \frac{1}{k+l} \sum_{i,j} \tilde{\phi}_{i,j}(d_{i,j})$ on $M_{k+l}(W_x)$ admits a $C(X)$-linear unital positive extension to $M_{k+l}(W)$ ([6]). Thus, there is a $C(X)$-linear completely contractive map $\varphi : V \to M_{k,l}(C(X))$ with $\varphi_x = \phi$ and $\|\varphi\|_{cb} = \|\phi\|_{cb}$ ([25], [34]). ☐

We end this section with a short proof of the implication $\gamma_e) \Rightarrow \alpha_e)$ given in [22] *i.e.* the characterization of the exactness of a $C(X)$-algebra A by assertion γ_e, if the topological space X is *perfect*, i.e. without any isolated point.

$\gamma_e) \Rightarrow \alpha_e)$ Given two $C(X)$-algebras A, B and a point $x \in X$, we have canonical $*$-epimorphisms $q_x : A \overset{m}{\otimes} B \to (A_x \overset{m}{\otimes} B)_x$ and $q'_x : (A_x \overset{m}{\otimes} B)_x \to A_x \overset{m}{\otimes} B_x$. Further, $q_x(f \otimes 1 - 1 \otimes f) = f(x) - f(x) = 0$ for all $f \in C(X)$. Hence q_x factorizes through $A \otimes_{C(X)} B$ if the $C(X)$-algebra A is continuous, by [5, proposition 3.1]. If B is also continuous and A satisfies γ_e), then $(A \underset{C(X)}{\overset{m}{\otimes}} B)_x \cong (A_x \overset{m}{\otimes} B)_x \cong A_x \overset{m}{\otimes} B_x$ and so the $C(X)$-algebra $A_x \overset{m}{\otimes} B$ is continuous at x. Thus, Corollary 3 of [13] implies that each fibre A_x is exact ($x \in X$) and the equivalence between assertions (i) and (iv) in [22, Thm. 4.6] entails that the C*-algebra A itself is exact.

3. The full amalgamated free product

In this section, we study the continuity of the full free product amalgamated over $C(X)$ of two unital continuous $C(X)$-algebra ([26], [28]). By default all tensor products and free products will be over $\mathbb{C}$.

Definition 3.1. ([33]) Let X be a compact Hausdorff space and let A_1, A_2 be two unital $C(X)$-algebras containing a unital copy of $C(X)$ in their centres, *i.e.* $1_{A_i} \in C(X) \subset A_i$ $(i = 1, 2)$.
– The *algebraic free product of* A_1 *and* A_2 *with amalgamation over* $C(X)$ is the unital quotient $A_1 \underset{C(X)}{\circledast} A_2$ of the algebraic free product of A_1 and A_2 over $\mathbb{C}$ by the two sided ideal generated by the differences $f1_{A_1} - f1_{A_2}$, $f \in C(X)$.
– The *full amalgamated free product free product* $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ is the universal unital enveloping C*-algebra of the $*$-algebra $A_1 \underset{C(X)}{\circledast} A_2$.
– Any pair (σ_1, σ_2) of unital $*$-representations of A_1, A_2 that coincide on their restrictions to $C(X)$ defines a unital $*$-representation $\sigma_1 * \sigma_2$ of $A_1 \underset{C(X)}{\circledast} A_2$, the restriction of which to A_i coincides with σ_i $(i = 1, 2)$.

In particular, the two unital central copies of $C(X)$ in A_1 and A_2 coherently define a structure of $C(X)$-algebra on $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ and by universality, we have:

$$\forall\, x \in X\,, \quad (A_1 \underset{C(X)}{\overset{f}{*}} A_2)_x \cong (A_1)_x \underset{\mathbb{C}}{\overset{f}{*}} (A_2)_x\,. \tag{3.1}$$

Remark 3.2. If we fix unital positive $C(X)$-linear maps $\varphi_i : A_i \to C(X)$ and we set $A_i^\circ = \ker \varphi_i$ for $i = 1, 2$, then the algebraic amalgamated free product $A_1 \underset{C(X)}{\circledast} A_2$ is (isomorphic to) the $C(X)$-module $C(X) \oplus \underset{n \geq 1}{\oplus} \bigoplus_{i_1 \neq \ldots \neq i_n} A_{i_1}^\circ \underset{C(X)}{\otimes} \ldots \underset{C(X)}{\otimes} A_{i_n}^\circ$, which is a $*$-algebra for the product $v.w := v \otimes w$ and the involution $(v.w)^* = w^*.v^*$.

Assume now that A_1 and A_2 are continuous $C(X)$-algebras. Then $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ is also a continuous $C(X)$-algebra as soon as both A_1 and A_2 are separable exact C*-algebras thanks to an embedding property due to Pedersen ([26]):

Proposition 3.3. *Let X be a compact Hausdorff space and A_1, A_2 two separable unital continuous $C(X)$-algebras which are exact C*-algebras.*

Then the full amalgamated free product $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ is a continuous $C(X)$-algebra.

Proof. For $i = 1, 2$, let π_i be a $C(X)$-linear embedding of A_i into $C(X; \mathcal{O}_2)$ (§2.3). Then the induced $C(X)$-linear morphism $\pi_1 * \pi_2$ from $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ to the continuous $C(X)$-algebra $C(X; \mathcal{O}_2) \underset{C(X)}{\overset{f}{*}} C(X; \mathcal{O}_2) = C(X; \mathcal{O}_2 \overset{f}{*} \mathcal{O}_2)$, is injective by [26, Thm. 4.2]. □

This continuity property actually always holds (Theorem 3.7). In order to prove it, let us first state the following Lemma which will enable us to reduce the problem to the separable case. (Its proof is the same as in [10, 2.4.7].)

Lemma 3.4. *Let X be a compact Hausdorff space, A_1, A_2 two unital $C(X)$-algebras and a an element of the algebraic amalgamated free product $A_1 \underset{C(X)}{\circledast} A_2$. Then there exist a second countable compact Hausdorff space Y s.t. $1_{C(X)} \in C(Y) \subset C(X)$ and separable $C(Y)$-algebras $D_1 \subset A_1$ and $D_2 \subset A_2$ s.t. a belongs to the $*$-subalgebra $D_1 \underset{C(Y)}{\circledast} D_2$.*

Let $e_1, e_2, \ldots$ be an orthonormal basis of $\ell^2(\mathbb{N})$ and set $e_{i,j} := \theta_{e_i, e_j}$ for all i, j in $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ (see (2.1)) Note that $e_{i,j}$ is the rank 1 partial isometry such that $e_{i,j} e_j = e_i$. Then the following critical Lemma holds:

Lemma 3.5. *Let Y be a second countable compact space, D_1, D_2 two separable unital continuous $C(Y)$-algebras and set $\mathcal{D} = D_1 \underset{C(Y)}{\overset{f}{*}} D_2$.*

If the element d belongs to the algebraic amalgamated free product $D_1 \underset{C(Y)}{\circledast} D_2$, then the function $y \mapsto \|d_y\|_{\mathcal{D}_y}$ is continuous.

Proof. The map $y \mapsto \|d_y\|$ is always upper semicontinuous by (3.1). So, it only remains to prove that it is also lower semicontinuous if both the $C(Y)$-algebras D_1 and D_2 are continuous.

Now, any element d in the algebraic amalgamated free product of D_1 and D_2 admits by construction (at least) one finite decomposition (that we fix) in $M_m(\mathbb{C}) \otimes \mathcal{D}$

$$e_{1,1} \otimes d = d(1) \ldots d(2n) \tag{3.2}$$

for suitable integers $m, n \in \mathbb{N}$ and elements $d(k) \in M_m(\mathbb{C}) \otimes D_{\iota_k}$ $(1 \leq k \leq 2n)$, where $\iota_k = 1$ if k is odd and $\iota_k = 2$ otherwise.

Given a point $y \in Y$ and a constant $\varepsilon > 0$, there exist unital $*$-representations Θ_1, Θ_2 of the fibres $(D_1)_y, (D_2)_y$ on $\ell^2(\mathbb{N})$ and unit vectors ξ, ξ' in $e_1 \otimes \ell^2(\mathbb{N}) \subset \mathbb{C}^m \otimes \ell^2(\mathbb{N})$ s.t.

$$\|d_y\| - \varepsilon < \Big| \langle \xi', \left[e_{1,1} \otimes (\Theta_1 * \Theta_2)(d_y) \right] \xi \rangle \Big| . \tag{3.3}$$

As the sequence of projections $p_k = \sum_{i=0}^{k} e_{i,i}$ in $\mathcal{K}(\ell^2(\mathbb{N}))$ satisfies $\lim\limits_{k \to \infty} \|(1 - p_k)\zeta\| = 0$ for all $\zeta \in \ell^2(\mathbb{N})$, a finite induction implies that there is an integer $l \in \mathbb{N}$ such that $\|(1 \otimes p_l)\xi\| \neq 0$, $\|(1 \otimes p_l)\xi'\| \neq 0$ and the two u.c.p. maps $\phi_i(\,.\,) = p_l \Theta_i(\,.\,) p_l$ on the fibres $(D_i)_y$ $(i = 1, 2)$ satisfy:

$$\left| \langle \xi'_l, \Big[e_{1,1} \otimes (\Theta_1 * \Theta_2)(d_y) - (id \otimes \phi_{\iota_1})(d(1)_y) \ldots (id \otimes \phi_{\iota_{2n}})(d(2n)_y) \Big] \xi_l \rangle \right| < \varepsilon \tag{3.4}$$

where $\xi_l = (1 \otimes p_l)\xi / \|(1 \otimes p_l)\xi\|$ and $\xi'_l = (1 \otimes p_l)\xi / \|(1 \otimes p_l)\xi'\|$ are unit vectors in $\mathbb{C}^m \otimes \mathbb{C}^{l+1}$ which are arbitrarily close to ξ and ξ', respectively, for sufficiently large l.

Let $\zeta_l \in \mathbb{C}^l \otimes \mathbb{C}^l$ be the unit vector $\zeta_l = \frac{1}{\sqrt{l}} \sum_{1 \leq k \leq l} e_k \otimes e_k$. For each i, the state $e \mapsto \langle \zeta_l, (id \otimes \phi_i)(e)\, \zeta_l \rangle$ on $M_l(\mathbb{C}) \otimes (D_i)_y$ associated to ϕ_i admits a unital $C(Y)$-linear positive extension $\Psi_i : M_l(\mathbb{C}) \otimes D_i \to C(Y)$ ([6]). If $(\mathcal{H}_i, \eta_i, \sigma_i)$ is the associated GNS-Kasparov construction, then every $d_i \in D_i$ satisfies

$$\langle 1_l \otimes \eta_i, (id \otimes \sigma_i)(\theta_{\zeta_l, \zeta_l} \otimes d_i) 1_l \otimes \eta_i \rangle (y) = (id \otimes \Psi_i)(\theta_{\zeta_l, \zeta_l} \otimes d_i)(y) = \phi_i((d_i)_y) \quad (3.5)$$

Let $\sigma = \sigma_1 * \sigma_2$ be the $*$-representation of the full amalgamated free product $\mathcal{D}$ on the amalgamated pointed free product $C(Y)$-module $(\mathcal{H}, \eta) = *_{C(Y)}(\mathcal{H}_i, \eta_i)$ ([33]). Then

$$\begin{aligned} \left| \langle \xi'_l \otimes \eta, e_{1,1} \otimes \sigma(d)\, \xi_l \otimes \eta \rangle \right| (y) &= \left| \langle \xi'_l \otimes \eta, (id \otimes \sigma)(d(1)) \ldots (id \otimes \sigma)(d(2n)) \xi_l \otimes \eta \rangle \right| (y) \\ &= \left| \langle \xi'_l, (id \otimes \phi_{\iota_1})(d(1)_y) \ldots (id \otimes \phi_{\iota_{2n}})(d(2n)_y) \xi_l \rangle \right| \\ &> \|d_y\| - 2\varepsilon \end{aligned}$$

And so, $\|d_y\| - 2\varepsilon < \left| \langle \xi'_l \otimes \eta, e_{1,1} \otimes \sigma(d)\, (1 \otimes p_l) \xi_l \otimes \eta \rangle \right| (z) \leq \|d_z\|$ for all point z in an open neighbourhood of y in Y by continuity. □

Remark 3.6. The referee pointed out that the inequality (3.4) cannot be replaced by a norm inequality like $\left\| e_{1,1} \otimes (\Theta_1 * \Theta_2)(d_y) - (id \otimes \phi_{\iota_1})(d(1)_y) \ldots (id \otimes \phi_{\iota_{2l}})(d(2l)_y) \right\| < \varepsilon'$. Indeed, if for instance $p \in A = M_2(\mathbb{C})$ is the projection $p = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $e_{1,1} \,.\, e_{2,2} = 0 \quad$ but $\quad p\, e_{1,1}\, p\, e_{2,2}\, p = 2^{-3/2} p \neq 0\,.$

Theorem 3.7. *Let X be a compact Hausdorff space and let A_1, A_2 be two unital continuous $C(X)$-algebras. Then the full amalgamated free product $\mathcal{A} = A_1 \underset{C(X)}{\overset{f}{*}} A_2$ is a continuous $C(X)$-algebra with fibres $\mathcal{A}_x = (A_1)_x \overset{f}{*} (A_2)_x$ ($x \in X$).*

Proof. The $C(X)$-algebra $\mathcal{A}$ has fibre $\mathcal{A}_x = (A_1)_x \overset{f}{*} (A_2)_x$ at $x \in X$ by (3.1). Hence it is enough to prove that for all a in the dense algebraic amalgamated free product $A_1 \underset{C(X)}{\circledast} A_2 \subset \mathcal{A}$, the map $x \mapsto \|a_x\|$ is lower semi-continuous.

Let a be such an element and choose a finite decomposition $e_{1,1} \otimes a = a_1 \ldots a_{2n} \in M_m(\mathbb{C}) \otimes \mathcal{A}$, where $m, n \in \mathbb{N}$ and a_k belongs to $M_m(\mathbb{C}) \otimes A_1$ or $M_m(\mathbb{C}) \otimes A_2$ according to the parity of k. By Lemma 3.4, there exist a separable unital C*-subalgebra $C(Y) \subset C(X)$ containing the unit $1_{C(X)}$ of $C(X)$ and separable unital C*-subalgebras $D_1 \subset A_1$ and $D_2 \subset A_2$ such that each D_i is a continuous $C(Y)$-algebra and all the a_k belong to $M_m(\mathbb{C}) \otimes D_1$ or $M_m(\mathbb{C}) \otimes D_2$ according to the parity of k. And so a also belongs to the full free product $\mathcal{D} = D_1 \underset{C(Y)}{\overset{f}{*}} D_2$ which admits a $C(Y)$-linear embedding in $A_1 \underset{C(X)}{\overset{f}{*}} A_2$ ([26, Thm. 4.2]). Hence it is enough to prove that the map $y \in Y \mapsto \|a_y\|_{\mathcal{D}_y}$ is also lower semicontinuous. But this follows from Lemma 3.5. □

4. The reduced amalgamated free product

Let us now study the continuity properties of certain reduced amalgamated free product over $C(X)$ of two unital continuous $C(X)$-algebras ([32], [33]).

The main result of this section is the following:

Theorem 4.1. *Let X be a compact Hausdorff space and let A_1, A_2 be two unital continuous $C(X)$-algebras. For $i = 1, 2$, let $\phi_i : A_i \to C(X)$ be a unital projection such that for all $x \in X$, the induced state $(\phi_i)_x$ on the fibre $(A_i)_x$ has faithful GNS representation.*

If the C-algebra A_1 is exact, then the reduced amalgamated free product*

$$(A, \phi) = (A_1, \phi_1) \underset{C(X)}{*} (A_2, \phi_2)$$

is a continuous $C(X)$-algebra with fibres $(A_x, \phi_x) = ((A_1)_x, (\phi_1)_x) * ((A_2)_x, (\phi_2)_x)$.

The proof is similar to the one used by Dykema and Shlyakhtenko in [17, §4] to prove that a reduced free product of exact C*-algebras is exact. We shall accordingly omit details except where our proof deviates from theirs.

Lemma 4.2. *Let A be a $C(X)$-algebra and $J \triangleleft A$ be a closed two sided ideal in A. If the two $C(X)$-algebras J and A/J are continuous, then A is also continuous.*

Proof. The canonical $C(X)$-linear representation π of A on $J \oplus A/J$ is a continuous field of faithful representations. Indeed, if $a \in A$ satisfies $\pi_x(a_x) = 0$ for some $x \in X$, then $(aa' + J)_x = 0$ for all $a' \in A$, hence $(a + J)_x = 0$, *i.e.* $a_x \in J_x$. Now $a_x h_x = (ah)_x = 0$ for all $h \in J$ and so $a_x = 0$. □

Remark 4.3. The continuity of the $C(X)$-algebra A does not imply the continuity of the quotient A/J. In fact, any $C(X)$-algebra B is the quotient of the constant $C(X)$-algebra $A = C(X; B) = C(X) \otimes B$ by the (closed) two sided ideal $C_\Delta.A$, where $C_\Delta \subset C(X \times X)$ is the ideal of functions f which satisfy $f(x, x) = 0$ for all $x \in X$.

Lemma 4.4. *Let B be a unital $C(X)$-algebra and E a full countably generated Hilbert B-module. Then B is a continuous $C(X)$-algebra if and only if the $C(X)$-algebra $\mathcal{K}_B(E)$ of compact operators acting on E (Definition 2.1) is continuous.*

Proof. The C*-algebra B and $\mathcal{K}_B(E)$ are stably isomorphic by Kasparov stabilisation theorem ([4, Thm. 13.6.2]), *i.e.* there is a B-linear isomorphism

$\mathcal{K}(\ell^2(\mathbb{N})) \otimes B \cong \mathcal{K}_B(\ell^2(\mathbb{N}) \otimes E)) \cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}_B(E)$ ([4, Ex. 13.7.1]).

Note that this isomorphism are also $C(X)$-linear since $1_B \in C(X) \subset B$. As the C*-algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ is nuclear, the Theorem 3.2 of [22] implies the equivalence between the continuity of the $C(X)$-algebras B, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes B$ and $\mathcal{K}_B(E)$. □

Given a C*-algebra B and a Hilbert B-bimodule E, recall that the full Fock Hilbert B-bimodule associated to E is the sum $\mathscr{F}_B(E) = B \oplus E \oplus (E \otimes_B E) \oplus \ldots = \bigoplus_{n \in \mathbb{N}} E^{(\otimes_B)n}$ and that for all $\zeta \in E$, the creation operator $\ell(\zeta) \in \mathcal{L}_B(\mathscr{F}_B(E))$ is defined by

$$\begin{aligned} &\bullet\ \ \ell(\zeta)b = \zeta b && \text{for } b \in B =: E^0 \quad \text{and} \\ &\bullet\ \ \ell(\zeta)(\zeta_1 \otimes \ldots \otimes \zeta_k) = \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k && \text{for } \zeta_1, \ldots, \zeta_k \in E\,. \end{aligned} \tag{4.1}$$

Then the Toeplitz C*-algebra $\mathfrak{T}_B(E)$ of the Hilbert B-module E (called *extended Cuntz-Pimsner algebra* in [17]) is the C*-subalgebra of $\mathcal{L}_B(\mathscr{F}_B(E))$ generated by the operators $\ell(\zeta)$, $\zeta \in E$ ([27]).

Lemma 4.5. *Let B be a unital $C(X)$-algebra and let E be a countably generated Hilbert B-bimodule such that the left module map $B \to \mathcal{L}_B(E)$ is injective and satisfies*

$$f.\zeta = \zeta.f \quad \text{for all } \zeta \in E \text{ and } f \in C(X)\,. \tag{4.2}$$

Then the Toeplitz $C(X)$-algebra $\mathfrak{T}_B(E)$ of E is a continuous $C(X)$-algebra with fibres $\mathfrak{T}_{B_x}(E_x)$ if and only if the $C(X)$-algebra B is continuous.

Under the assumption of this Lemma, the canonical $*$-monomorphism $B \to \Pi_{x\in X} B_x$ induces for any Hilbert B-module F a $*$-homomorphism $a \mapsto a\otimes 1$ from the C*-algebra $\mathcal{K}_B(F)$ of compact operators acting on the Hilbert module F to the tensor product $\mathcal{K}_B(F) \otimes_B (\Pi_{x\in X} B_x) \cong \Pi_{x\in X} \mathcal{K}_{B_x}(F \otimes_B B_x)$. And this map is injective as soon as there is a B-linear decomposition $F \cong B \oplus F'$ for some Hilbert B-module F'. After passing to the multiplier C*-algebras, this gives for $F = \mathscr{F}_B(E)$ a $*$-monomorphism $\Theta = (\Theta_x)$:

$$\mathcal{L}_B(\mathscr{F}_B(E)) = \mathcal{M}\big(\mathcal{K}_B(\mathscr{F}_B(E))\big) \hookrightarrow \prod_{x\in X} \mathcal{M}\big(\mathcal{K}_{B_x}(\mathscr{F}_{B_x}(E_x))\big) = \prod_{x\in X} \mathcal{L}_{B_x}(\mathscr{F}_{B_x}(E_x))\,,$$

where E_x is the Hilbert B_x-module $E_x = E \otimes_B B_x \cong E/C_x(X)E$ for all $x \in X$.

Note that for all $\zeta \in E$ and $x \in X$, we have $\ell(\zeta)\, C_x(X)\mathscr{F}_B(E) \subset C_x(X)\mathscr{F}_B(E)$ by (4.2) and so the element $\Theta_x(\ell(\zeta))$ satisfies the same creation rules (4.1) as the creation operator $\ell(\zeta_x)$, where $\zeta_x = \zeta \otimes 1_{B_x} \in E_x$. So, the restriction of Θ to $\mathfrak{T}_B(E)$ takes values in the product $\prod_{x\in X} \mathfrak{T}_{B_x}(E_x)$.

Proof of Lemma 4.5. The continuity of the $C(X)$-algebra $\mathfrak{T}_B(E)$ clearly implies the continuity of the $C(X)$-algebra B since B embeds $C(X)$-linearly in $\mathfrak{T}_B(E)$.

Suppose conversely that the $C(X)$-algebra B is continuous. Let $\widetilde{E}$ be the full countably generated Hilbert B-bimodule $\widetilde{E} = E \oplus B$. Then the $C(X)$-algebra $\mathcal{K}_B(\mathscr{F}_B(\widetilde{E}))$ of compact operators acting on the Hilbert B-module $\mathscr{F}_B(\widetilde{E})$ is a continuous $C(X)$-algebra by Lemma 4.4. Hence it is enough to prove that the Toeplitz C*-algebra $\mathfrak{T}_B(\widetilde{E})$ admits a continuous field of faithful representations $\mathfrak{T}_B(\widetilde{E}) \to \mathcal{L}_B(\mathscr{F}_B(\widetilde{E})) = \mathcal{M}(\mathcal{K}_B(\mathscr{F}_B(\widetilde{E})))$ since $\mathfrak{T}_B(E)$ embeds in $\mathfrak{T}_B(\widetilde{E}) = \mathfrak{T}_B(E \oplus B)$ ([27] or [17, §4]).

Step 1. Let β be the action of the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on $\mathfrak{T}_B(\widetilde{E})$ determined by $\beta_t(\ell(\zeta)) = e^{2i\pi t}\ell(\zeta)$ for $\zeta \in \widetilde{E}$. Then the fixed point C-subalgebra A under this action is a continuous $C(X)$-algebra and the map $A_x \to \mathfrak{T}_{B_x}(\widetilde{E}_x)$ is injective for all $x \in X$.*

Define the increasing sequence of B-subalgebra $A_n \subset A$ generated by the words of the form $w = \ell(\zeta_1)\ldots\ell(\zeta_k)\,\ell(\zeta_{k+1})^*\ldots\ell(\zeta_{2k})^*$ with $k \le n$. Let also $A_0 = B$. As $A = \overline{\cup A_n}$, it is enough to prove that each A_n is continuous with appropriate fibres.

Let $\widetilde{E}_0 = B$ and $\widetilde{E}_n = \widetilde{E} \otimes_B \ldots \otimes_B \widetilde{E} = \widetilde{E}^{(\otimes_B)n}$ for $n \ge 1$. Define also the projection $P_n \in \mathcal{L}_B(\mathscr{F}_B(\widetilde{E}))$ on the Hilbert B-bimodule $F_n = \oplus_{0\le k\le n}\widetilde{E}_k$. Then the $C(X)$-linear completely positive map $a \in A_n \mapsto P_n a P_n$ is faithful. Further, the kernel of the composition of that map with the restriction map $F_n = F_{n-1} \oplus \widetilde{E}_n \to F_{n-1}$ is the B-module generated by the words $w = \ell(\zeta_1)\dots\ell(\zeta_n)\,\ell(\zeta_{n+1})^*\dots\ell(\zeta_{2n})^*$ of length $2n$, which isomorphic to $\mathcal{K}_B(\widetilde{E}_n)$. Hence, we have by induction $C(X)$-linear split exact sequences

$$0 \to \mathcal{K}_B(\widetilde{E}_n) \to A_n \to A_{n-1} \to 0\,, \tag{4.3}$$

and so, each $C(X)$-algebra A_n is continuous by lemma 4.2.

Step 2. The Toeplitz C-algebra $\mathfrak{T}_B(\widetilde{E})$ is isomorphic to the crossed product $A \rtimes_\alpha \mathbb{N}$, where $\alpha : A \to A$ is the injective $C(X)$-linear endomorphism $\alpha(a) = LaL^*$, with $L = \ell(0 \oplus 1_B)$ ([17, Claim 3.3]). Hence $\mathfrak{T}_B(\widetilde{E})$ is a continuous field with fibres $\big(\mathfrak{T}_B(\widetilde{E})\big)_x \cong A_x \rtimes \mathbb{N} \cong \mathfrak{T}_{B_x}(\widetilde{E}_x)$ for $x \in X$.*

The C*-algebra $\mathfrak{T}_B(\widetilde{E})$ is generated by A and L. Hence it is isomorphic to the $C(X)$-algebra $A \rtimes_\alpha \mathbb{N}$ ([17, claim 3.4]). Let us now study the continuity question.

Let $\ddot{A}$ be the inductive limit of the system $A \overset{\alpha}{\to} A \overset{\alpha}{\to} \dots$ with corresponding $C(X)$-linear monomorphisms $\mu_n : A \to \ddot{A}$ ($n \in \mathbb{N}$). It is a continuous $C(X)$-algebra since $\bigcup_n \mu_n(A)$ is dense in $\ddot{A}$ and the map $x \in X \mapsto \|\mu_n(a)_x\| = \|a_x\|$ is continuous for all $(a, n) \in A \times \mathbb{N}$. Let $\ddot{\alpha} : \ddot{A} \to \ddot{A}$ be the $C(X)$-linear automorphism given by $\ddot{\alpha}\big(\mu_n(a)\big) = \mu_n\big(\alpha(a)\big)$, with inverse $\mu_n(a) \mapsto \mu_{n+1}(a)$. Then the crossed product $\ddot{A} \rtimes_{\ddot{\alpha}} \mathbb{Z}$ is continuous over X since the group $\mathbb{Z}$ is amenable ([30]). Hence, if $p \in \ddot{A}$ is the projection $p = \mu_0(1_A)$, the hereditary $C(X)$-subalgebra $p\left(\ddot{A} \rtimes_{\ddot{\alpha}} \mathbb{Z}\right)p =: A \rtimes_\alpha \mathbb{N}$ ([17]) is also continuous with fibres $A_x \rtimes_{\alpha_x} \mathbb{N}$ ($x \in X$). □

Proof of Theorem 4.1. By density, it is enough to study the case of elements a in the algebraic amalgamated free product $A_1 \underset{C(X)}{\circledast} A_2$. But, for any such a, there are separable unital C*-subalgebras $C(Y) \subset C(X)$, $D_1 \subset A_1$ and $D_2 \subset A_2$ with same units such that a also belongs to $D_1 \underset{C(Y)}{\circledast} D_2$ by Lemma 3.4. And the reduced free product $(D_1, \phi_1) \underset{C(Y)}{*} (D_2, \phi_2)$ embeds $C(X)$-linearly in $(A, \phi) = (A_1, \phi_1) \underset{C(X)}{*} (A_2, \phi_2)$ by [9, Thm. 1.3]. Further, any C*-subalgebra of an exact C*-algebra is exact.Thus, one can assume in the sequel that the compact Hausdorff space X is second countable and that the $C(X)$-algebras A_1, A_2 are separable C*-algebras.

If the C*-algebra A_1 is exact, then the $C(X)$-algebra $B = A_1 \underset{C(X)}{\otimes} A_2$ is continuous with fibres $B_x = (A_1)_x \overset{m}{\otimes} (A_2)_x$ by γ_e), and the conditional expectation $\rho = \phi_1 \otimes \phi_2 : B \to C(X)$ is a continuous field of states on B such that each ρ_x has faithful GNS representation ($x \in X$).

Let E be the full countably generated Hilbert B,B–bimodule $L^2(B, \rho) \otimes_{C(X)} B$ and let $\mathscr{F}_B(E) = B \oplus \left(L^2(B, \rho) \underset{C(X)}{\otimes} B\right) \oplus \left(L^2(B, \rho) \underset{C(X)}{\otimes} L^2(B, \rho) \underset{C(X)}{\otimes} B\right) \oplus \dots$ be its full Fock bimodule. Let also $\xi = \Lambda_\phi(1) \otimes 1 \in E$. As observed in [17, Claim 3.3], the Toeplitz C*-algebra $\mathfrak{T}_B(E) \subset \mathcal{L}_B(\mathscr{F}_B(E))$ is generated by the left action of B on $\mathscr{F}_B(E)$ and the operator $\ell(\xi)$, because $\ell(b_1 \xi b_2) = b_1 \ell(\xi) b_2$ for all b_1, b_2 in B.

Consider the conditional expectation $\mathfrak{E} : \mathfrak{T}_B(E) \to B$ defined by compression with the orthogonal projection from $\mathscr{F}_B(E)$ into the first summand $B \subset \mathscr{F}_B(E)$. Then Theorem 2.3 of [29] implies that B and the $C(X)$-algebra generated by the non trivial isometry $\ell(\xi)$ are free with amalgamation over $C(X)$ in $(\mathfrak{T}_B(E), \rho \circ \mathfrak{E})$ because $\ell(\xi)^* \, b \, \ell(\xi) = \rho(b)$ for all $b \in B$.

By [29], the restriction of $\mathfrak{E}$ to the C*-subalgebra $C^*(\ell(\xi)) \subset \mathfrak{T}_B(E)$ takes values in $C(X)$ and there exists a unitary $u \in C^*(\ell(\xi))$ s.t. $\mathfrak{E}(u^k) = 0$ for every non-zero integer k. The two embeddings $\pi_i : A_i \to \mathfrak{T}_B(E)$ $(i = 1, 2)$ given by $\pi_1(a_1) = u\,(a_1 \otimes 1)\,u^{-1}$ and $\pi_2(a_2) = u^2\,(1 \otimes a_2)\,u^{-2}$ have free images in $(\mathfrak{T}_B(E), \rho \circ \mathfrak{E})$. Thus they generate a $C(X)$-linear monomorphism $\pi : A \hookrightarrow \mathfrak{T}_B(E)$ extending each π_k and satisfying $\rho \circ \mathfrak{E} \circ \pi = \phi$ (Lemma 4.1 and Proposition 4.2 of [17], or [9]). Above Lemma 4.5 entails that A is a continuous $C(X)$-algebra with fibre at $x \in X$ its image in $\mathfrak{T}_{B_x}(E_x)$, i.e. the reduced free product $((A_1)_x, (\phi_1)_x) * ((A_2)_x, (\phi_2)_x)$. □

Remark 4.6. The existence of an embedding of A_1 in $C(X; \mathcal{O}_2)$ cannot give a direct proof of Theorem 4.1 since there is no $C(X)$-linear Hahn-Banach theorem ([7, 4.2]).

Corollary 4.7. *Any separable unital continuous $C(X)$-algebra A admits a $C(X)$-linear unital embedding into a unital continuous field $\widetilde{A}$ with simple fibres.*

Proof. Let $\phi : A \to C(X)$ be a $C(X)$-linear unital map such that each induced state $\phi_x : A_x \to \mathbb{C}$ is faithful. Then, the reduced free product

$$(\widetilde{A}, \Phi) = (A \otimes \mathbb{C}^2; \phi \otimes tr_2) *_{C(X)} (C(X) \otimes \mathbb{C}^3; \mathrm{id} \otimes tr_3)$$

is continuous by Proposition 4.1, and it has simple fibres ([2], [3]). □

Corollary 4.8. *Let X be a second countable perfect compact space and A_1 a unital separable continuous $C(X)$-algebra. Then the following assertions are equivalent.*

- α) *The C*-algebra A_1 is exact.*
- β) *For all unital separable continuous $C(X)$-algebra A_2 and all continuous fields of faithful states ϕ_1 and ϕ_2 on A_1 and A_2, the reduced amalgamated free product $(A, \phi) = (A_1, \phi_1) \underset{C(X)}{*} (A_2, \phi_2)$ is a continuous $C(X)$-algebra with fibres $(A_x, \phi_x) = ((A_1)_x, (\phi_1)_x) * ((A_2)_x, (\phi_2)_x)$.*

Proof. We only need to prove the implication $\beta) \Rightarrow \alpha)$ since the reverse implication has already been proved in Theorem 4.1.

Now, if a pair (A_2, ϕ_2) satisfies the hypotheses of β) and we define the $C(X)$-algebra $B := A_1 \otimes_{C(X)} A_2$, the $C(X)$-linear projection $\rho = \phi_1 \otimes \phi_2 : B \to C(X)$ and the Hilbert B-module $E = L^2(B, \rho) \otimes_{C(X)} B$, then we have a $C(X)$-linear isomorphism $A \rtimes_\alpha \mathbb{N} \cong \mathcal{T}_B(E \oplus B)$ (Step 2 of Lemma 4.5). Hence, the Toeplitz $C(X)$-algebra $\mathcal{T}_B(E \oplus B)$ is continuous since the group $\mathbb{Z}$ is amenable (see e.g. [30]). And so, the amalgamated tensor product $A_1 \otimes_{C(X)} A_2$ is a continuous $C(X)$-algebra for any unital separable continuous $C(X)$-algebra A_2 (Lemma 4.5). But this implies the exactness of the C*-algebra A_1 if the metrizable space X is perfect ([11, Theorem 1.1]). □

Remark 4.9. Corollary 4.8 does not always hold if the space X is not perfect. For instance, if X is reduced to a point, then the reduced amalgamated free product of A_1 and A_2 is always continuous.

References

[1] R. J. Archbold, *Continuous bundles of C^*-algebras and tensor products*, Quart. J. Math. Oxford **50** (1999), 131–146.

[2] D. Avitzour, *Free products of C^*-algebras*, Trans. Amer. Math. Soc. **271** (1982), 423–465.

[3] B. Barnett, *Free product von Neumann algebras of type III*, Proc. Amer. Math. Soc. **123** (1995), 543–553.

[4] B. Blackadar, *K-theory for Operator Algebras*, MSRI Publications **5**, Cambridge Univ. Press (1998).

[5] E. Blanchard, *Tensor products of $C(X)$-algebras over $C(X)$*, Astérisque **232** (1995), 81–92.

[6] E. Blanchard, *Déformations de C^*-algèbres de Hopf.* Bull. Soc. Math. France, **24** (1996), 141–215.

[7] E. Blanchard, *Subtriviality of continuous fields of nuclear C^*-algebras.* J. Reine Angew. Math. **489** (1997), 133–149.

[8] E. Blanchard, *A few remarks on C(X)-algebras*, Rev. Roumaine Math. Pures Appl. **45** (2000), 565–576.

[9] E. Blanchard, K. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. **199** (2001), 1–19.

[10] E. Blanchard, E. Kirchberg, *Global Glimm halving for C^*-bundles*, J. Op. Th. **52** (2004), 385–420.

[11] E. Blanchard, S. Wassermann, *Exact C^*-bundles*, Houston J. Math. **33** (2007), 1147–1159.

[12] D. Blecher, P. Muhly, V. Paulsen, *Categories of operator modules (Morita equivalence and projective modules)*, Mem. Amer. Math. Soc. **143** (2000).

[13] S. Catterall, S. Wassermann, *Continuous bundles of C^*-algebras with discontinuous tensor products*, Bull. London Math. Soc. **36** (2006), 647–656.

[14] M.-D. Choi, E.G. Effros, *Separable nuclear C^*-algebras and injectivity.* Duke Math. J. **43** (1976), 309–322.

[15] J. Dixmier, A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[16] K. Dykema, *Simplicity and the stable rank of some free product C^*-algebras*, Trans. Amer. Math. Soc. **351** (1999), 1-40.

[17] K. Dykema, D. Shlyakhtenko, *Exactness of Cuntz-Pimsner C^*-algebras*, Proc. Edinb. Math. Soc. **44** (2001), 425–444.

[18] E.G. Effros, E.C. Lance, *Tensor products of operator algebras*, Advances in Math **25** (1977), 1–34.

[19] E.G. Effros, Z.-J. Ruan, *Operator spaces.* London Mathematical Society Monographs. New Series, **23** (2000).

[20] G.A. Elliott, T. Natsume, R. Nest, *The Heisenberg group and K-theory.* K-Theory **7** (1993), 409–428.

[21] T. Giordano, J.A. Mingo, *Tensor products of C^*-algebras over abelian subalgebras.* J. London Math. Soc. **55** (1997), 170–180.

[22] E. Kirchberg, S. Wassermann, *Operations on continuous bundles of C^*-algebras.* Math. Ann. **303** (1995), 677–697.

[23] B. Magajna, *Module states and tensor products over abelian C^*-algebras.* Rev. Roumaine Math. Pures Appl. **44** (1999), 85–95

[24] N. Ozawa, *A non-extendable bounded linear map between C^*-algebras.* Proc. Edinb. Math. Soc. **44** (2001), 241–248.

[25] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press (2002).

[26] G.K. Pedersen, *Pullback and pushout constructions in C^*-algebra theory*, J. Funct. Anal. **167** (1999), 243–344.

[27] M. V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun. **12** (1997), 189–212.

[28] G. Pisier, *Introduction to Operator space Theory.* London Math. Soc. Lecture Note Series **294**, Cambridge University Press (2003).

[29] D. Shlyakhtenko, *Some application of freeness with amalgamation*, J. reine angew, Math. **500** (1998), 191–212.
[30] M.A. Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions*, Math. Ann. **283** (1989), 631–643.
[31] Y. Ueda, *Remarks on free products with respect to non-tracial states*, Math. Scand. **88** (2001), 111–125.
[32] D. V. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Lecture Notes in Math. **1132** (1985), 556-588.
[33] D.V. Voiculescu, K. J. Dykema, A. Nica *Free Random Variables*, CRM Monograph Series 1, Amer. Math. Soc. , Providence (1992).
[34] S. Wassermann, *Exact C^*-algebras and related topics*, Lecture Notes Series **19** (1994), GARC, Seoul National University.

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